UNIT # 01

US02 CMTH21

COMPLEX NUMBERS

A number of the form $z = x + iy$, is called complex number.

Here x and y both are real numbers.

Real part of z is denoted by **Re(z)** and imaginary part of z is denoted by **Im(z)**.

in $z = x + iy$

$Re(z) = x$ and

$Im(z) = y$.

Conjugate of z is denoted by $\bar{z}$ and is defined as $\bar{z} = x - iy$

Modulus of z is denoted by $|z|$ and is defined as $|z| = \sqrt{x^2+y^2}$

Argument of z is denoted by $\arg z$ and is defined as

$$\arg z = \tan^{-1}(y/x).$$

Polar form of z: is $z = r(\cos\theta + i\sin\theta)$

Comparing with $z = x + iy$, we get

$r\cos\theta = x$, $r\sin\theta = y$.

$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

and $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Note :

- $z = \cos\theta + i\sin\theta$, can be written as $z = \text{cis}\,\theta$
- $|z| = |\bar{z}|$
- $|z_1 z_2| = |z_1||z_2|$.
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$,
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $z\bar{z} = |z|^2$
- $z = x + iy$

  $\bar{z} = x - iy$

  $\Rightarrow x = \frac{z + \bar{z}}{2}$

  $y = \frac{z - \bar{z}}{2i}$

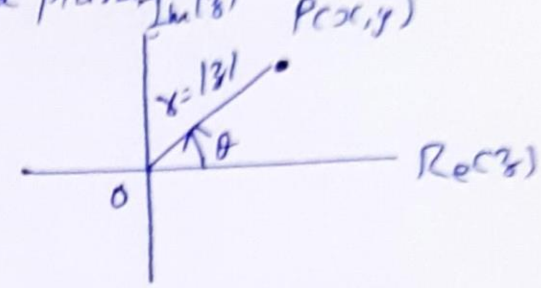
How to Find argument of a complex number ??

Let $z = x + iy$ is a complex no.

We wish to find argument of z

Let us denote no. $z = x + iy$ in argand plane

Point form of z is (x, y)



Distance of P from O is called $|z|$.

Angle θ formed by OP from Real axis is called argument.

- when θ is measured in anticlockwise direction it is positive and

  when θ is measured in clockwise direction it is negative.

For principal argument $-\pi < \theta \leq \pi$

- first of all write z in standard form.
- Write z in point form
- Find quadrant in which point lies. and follow following.

II $\theta = \pi - \alpha$	I $\theta = \alpha$
III $\theta = \alpha - \pi$	$\theta = -\alpha$ IV

To calculate α, we do $\tan\alpha = \left|\frac{y}{x}\right|$

and find θ as per above diagram.

Exp. Find argument of $z = -1 - i\sqrt{3}$

Solution
- Point form of z is $(-1, -\sqrt{3})$
- It lies in third quadrant.

We know, $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{-\sqrt{3}}{-1}\right|$

$= |\sqrt{3}|$

$= \sqrt{3}$

$\alpha = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

$\arg z = -\frac{2\pi}{3}$

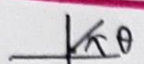
$0 < \theta \leq 2\pi$ $\quad -\pi < \theta \leq \pi$

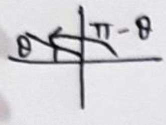
Exp: Find argument and Principal arguments of the following

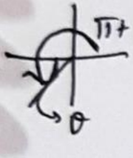
(i) $z = 2+2i$ (ii) $z = -3+3i$ (iii) $z = -1-\sqrt{3}i$ (iv) $1-\sqrt{3}i$

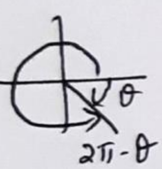
Solution.

Argument	Principal argument
(i) $z = 2+2i$ $\arg z = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1) = \pi/4$	$z = 2+2i$ $\arg z = \tan^{-1}\left\lvert\frac{2}{2}\right\rvert = \pi/4$
(ii) $z = -3+3i$ $\arg z = \pi - \tan^{-1}\left\lvert\frac{-3}{3}\right\rvert$ $= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$	$z = -3+3i$ $\arg z = \pi - \tan^{-1}\left\lvert\frac{3}{3}\right\rvert$ $= \pi - \pi/4 = 3\pi/4$
(iii) $z = -1-\sqrt{3}i$ $\arg z = \pi + \tan^{-1}\left\lvert\frac{\sqrt{3}}{1}\right\rvert$ $= \pi + \frac{\pi}{3} = \frac{4\pi}{3}$	$z = -1-\sqrt{3}i$ $\arg z = \tan^{-1}\left\lvert\frac{\sqrt{3}}{1}\right\rvert - \pi$ $= \frac{\pi}{3} - \pi = -\frac{2\pi}{3}$
(iv) $z = 1-\sqrt{3}i$ $\arg z = 2\pi - \tan^{-1}\left\lvert\frac{\sqrt{3}}{1}\right\rvert$ $= 2\pi - \frac{\pi}{3}$ $= \frac{5\pi}{3}$	$z = 1-\sqrt{3}i$ $\arg z = -\tan^{-1}\left\lvert\frac{\sqrt{3}}{1}\right\rvert$ $= -\pi/3$









Exp Find argument and modulus of the

$$z = \frac{(3-\sqrt{2}i)^2}{1+2i}$$

First of all we write z in standard form.

Sol.

$$z = \frac{(3-\sqrt{2}i)^2}{1+2i}$$

$$= \frac{(3-\sqrt{2}i)^2}{1+2i} \times \frac{1-2i}{1-2i}$$

$$z = \left(\frac{7-12\sqrt{2}}{5}\right) + i\left(\frac{-14-6\sqrt{2}}{5}\right)$$

$$\arg z = \tan^{-1}\left(\frac{-14-6\sqrt{2}}{7-12\sqrt{2}}\right)$$

Also $$|z| = \left[\left(\frac{7-12\sqrt{2}}{5}\right)^2 + \left(\frac{-14-6\sqrt{2}}{5}\right)^2\right]^{1/2}$$

$$= \frac{11}{\sqrt{5}} \quad \text{Ans}$$

Thm. (De-Moivre's Theorem)

State and prove De-Moivre's theorem

Proof: Statement is $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \forall n \in Q$

Case-1: n is a positive integer

$[cis\theta_1][cis\theta_2] = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$[cis\theta_1][cis\theta_2] = cis(\theta_1+\theta_2)$.

Similarly $[cis\theta_1][cis\theta_2][cis\theta_3] = cis(\theta_1+\theta_2+\theta_3)$

continuing in the same way, we get

$[cis\theta_1][cis\theta_2]\cdots[cis\theta_n] = cis(\theta_1+\theta_2+\cdots+\theta_n)$

If $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$, then

$[cis\theta][cis\theta]\cdots[cis\theta] = cis(\theta+\theta+\cdots+\theta)$

$(cis\theta)^n = cis\,n\theta. \quad \forall n \in N$

Case.2: If n is a negative integer.

Suppose $n = -m$, for all $m \in N$.

$(cis\theta)^n = (cis\theta)^{-m}$

$= \frac{1}{(cis\theta)^m} = \frac{1}{cis\,m\theta}$

$= \frac{1}{\cos m\theta + i\sin m\theta} \times \frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta}$

$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$

$= \cos m\theta - i\sin m\theta$

$= \cos(-m)\theta + i\sin(-m)\theta$

$= \cos n\theta + i\sin n\theta$

$= cis\,n\theta$, for all negative integers

Case.3 If $n = 0$

$(cis\theta)^0 = 1 = \cos 0 + i\sin 0$

Case-4 If $n = \frac{p}{q}$, $p \in \mathbb{Z}, q \in \mathbb{N}$.

$$(cis\theta)^n = (cis\theta)^{p/q}$$
$$= \left[(cis\theta)^{\frac{1}{q}}\right]^p \quad \text{--- (1)}$$

Now, $\left(cis\frac{\theta}{q}\right)^q = \left(cis\frac{\theta}{q}\cdot q\right) = cis\theta$

$$\Rightarrow cis\frac{\theta}{q} = (cis\theta)^{\frac{1}{q}}$$

From eq(1)

$$(cis\theta)^n = \left[cis\frac{\theta}{q}\right]^p = cis\left(\frac{\theta p}{q}\right)$$
$$= cis\,n\theta$$

Thus from cases 1, 2, 3 & 4, we can say

$$(cis\theta)^n = cis\,n\theta \quad \text{Proved.}$$

Exp. Prove that

$$(1+\cos\theta+i\sin\theta)^n + (1+\cos\theta-i\sin\theta)^n = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right).$$

Solution Let $1+\cos\theta = r\cos\alpha$

and $\sin\theta = r\sin\alpha$.

$$r = \sqrt{(1+\cos\theta)^2 + \sin^2\theta}$$
$$r = 2\cos\theta/2.$$
$$\alpha = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \tan^{-1}(\tan\theta/2)$$
$$\Rightarrow \alpha = \theta/2$$

$\therefore$ LHS.

$$(1+\cos\theta+i\sin\theta)^n + (1+\cos\theta-i\sin\theta)^n$$
$$= (r\cos\alpha + ir\sin\alpha)^n + (r\cos\alpha - ir\sin\alpha)^n$$
$$= r^n\left[(\cos\alpha+i\sin\alpha)^n + (\cos\alpha - i\sin\alpha)^n\right]$$
$$= r^n\left[\cos n\alpha + \cancel{i\sin n\alpha} + \cos n\alpha - \cancel{i\sin n\alpha}\right]$$
$$= 2r^n\cos n\alpha$$
$$= 2\left[2\cos\theta/2\right]^n \cos\frac{n\theta}{2} = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\frac{n\theta}{2} \quad \text{Proved}$$

(5)

Exp If $2\cos\theta = x + \frac{1}{x}$, then prove following

(i) $2\cos r\theta = x^r + \frac{1}{x^r}$

(ii) $\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\cos n\theta}{\cos(n-1)\theta}$.

Sol. Given $2\cos\theta = x + \frac{1}{x} \Rightarrow x^2 - 2x\cos\theta + 1 = 0$

$\Rightarrow x = \cos\theta \pm i\sin\theta$.

Consider $x = \cos\theta + i\sin\theta$

(i) RHS. $x^r + \frac{1}{x^r} = (\cos\theta + i\sin\theta)^r + (\cos\theta - i\sin\theta)^r$

$= 2\cos r\theta$

(ii) LHS. $\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{(\cos\theta+i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$

$= \frac{\cos n\theta}{\cos(n-1)\theta}$. Proved

Theorem. Prove that $(\text{cis}\,\theta)^{1/q} = \text{cis}\left(\frac{2n\pi+\theta}{q}\right)$, $n = 0, 1, 2, \dots q-1$

OR

Prove that there are q and only q distinct values of $((\cos\theta + i\sin\theta))^{1/q}$, where q is integer.

Proof We know that

$\text{cis}\,\theta = \text{cis}(2n\pi + \theta)$

$(\text{cis}\,\theta)^{1/q} = [\text{cis}(2n\pi+\theta)]^{1/q} = \text{cis}\left(\frac{2n\pi+\theta}{q}\right)$, $n = 0, 1, \dots q-1$

$n = 0$, $(\text{cis}\,\theta)^{1/q} = \text{cis}\left(\frac{\theta}{q}\right)$

$n = 1$, $(\text{cis}\,\theta)^{1/q} = \text{cis}\left(\frac{2\pi+\theta}{q}\right)$

$\vdots$

$n = q-1$, $(\text{cis}\,\theta)^{\frac{1}{q}} = \text{cis}\left(\frac{2(q-1)\pi+\theta}{q}\right)$

$n = q$, $(\text{cis}\,\theta)^{\frac{1}{q}} = \text{cis}\left(\frac{2q\pi+\theta}{q}\right) = \text{cis}\left(2\pi + \frac{\theta}{q}\right) = \text{cis}\left(\frac{\theta}{q}\right)$

Now values are repeating

$\therefore$ only distinct values are q.

Exp Find cube roots of unity. Also prove that they from an equilateral triangle in Argand diagram.

Sol. let $x = (1)^{1/3} \Rightarrow x^3 = 1 \Rightarrow x^3 - 1 = 0$.

$x = (1)^{1/3} = (\text{cis}\,0)^{1/3} = \text{cis}\left(\frac{2n\pi+0}{3}\right) = \text{cis}\left(\frac{2n\pi}{3}\right)$, $n = 0, 1, 2$

$x = \text{cis}\,0$, $x = \text{cis}\,\frac{2\pi}{3}$, $x = \text{cis}\,\frac{4\pi}{3}$

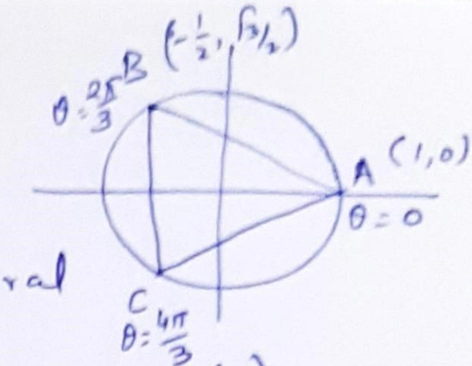
$= 1$ $\quad = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ $\quad = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Thus $x = 1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}-i\frac{\sqrt{3}}{2}$ are the cube roots of unity.

Here $AB = \sqrt{3}$, $BC = \sqrt{3}$, $AC = \sqrt{3}$

$\because$ $AB = BC = AC$

Hence ΔABC is an equilateral triangle.



Exp Find the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$. Also prove that the continued product of these values is 1.

Sol. Suppose $x = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4} = \left(\text{cis}\,\frac{\pi}{3}\right)^{3/4} = (\text{cis}\,\pi)^{1/4}$

$x = \text{cis}\left(\frac{2n\pi + \pi}{4}\right)$, $n = 0, 1, 2, 3$.

$n = 0$, $x = \text{cis}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}$

$n = 1$, $x = \text{cis}\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}$

$n = 2$, $x = \text{cis}\left(\frac{5\pi}{4}\right) = -\frac{1}{2} - i\frac{1}{\sqrt{2}}$

$n = 3$, $x = \text{cis}\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

Now the product of these values are $\left(\text{cis}\,\frac{\pi}{4}\right)\left(\text{cis}\,\frac{3\pi}{4}\right)\left(\text{cis}\,\frac{5\pi}{4}\right)\left(\text{cis}\,\frac{7\pi}{4}\right)$

$= \text{cis}\,4\pi = 1$. **Ans**

Exp Solve the equation.

$x^4 - x^3 + x^2 - x + 1 = 0$.

Sol. Here $x^4 - x^3 + x^2 - x + 1 = 0 \Rightarrow (x+1)(x^4 - x^3 + x^2 - x + 1) = 0$

$\Rightarrow x^5 + 1 = 0$

$\Rightarrow x = (-1)^{1/5} = (\text{cis}\,\pi)^{1/5}$

$x = \text{cis}\left(\frac{2n\pi + \pi}{5}\right)$, $n = 0, 1, 2, 3, 4$

$n = 0$, $x = \text{cis}\,\frac{\pi}{5}$

$n = 1$, $x = \text{cis}\,\frac{3\pi}{5}$

$n = 2$, $x = \text{cis}\,\pi = -1$

$n = 3$, $x = \text{cis}\,\frac{7\pi}{5}$

$n = 4$, $x = \text{cis}\,\frac{9\pi}{5}$

Hence the required roots are

$x = \text{cis}\,\frac{\pi}{5}$, $\text{cis}\,\frac{3\pi}{5}$, $\text{cis}\,\frac{7\pi}{5}$, $\text{cis}\,\frac{9\pi}{5}$

Exp Find the 7th root of unity.

Solution Let $x = (1)^{1/7} = (\text{cis}\,0)^{\frac{1}{7}} = \text{cis}\left(\frac{2n\pi}{7}\right)$, $n = 0, 1, \cdots 6$

$n = 0$, $x = \text{cis}\,0 = 1$

$n = 1$, $x = \text{cis}\,\frac{2\pi}{7}$

$n = 2$, $x = \text{cis}\,\frac{4\pi}{7}$

$n = 3$, $x = \text{cis}\,\frac{6\pi}{7}$

$n = 4$, $x = \text{cis}\,\frac{8\pi}{7}$

$n = 5$, $x = \text{cis}\,\frac{10\pi}{7}$

$c = 6$, $x = \text{cis}\,\frac{12\pi}{7}$

Ans

Exp: Find the equation whose roots are $2\cos\left(\frac{\pi}{7}\right), 2\cos\left(\frac{3\pi}{7}\right), 2\cos\left(\frac{5\pi}{7}\right)$.

Sol. Let $y = \text{cis}\,\theta$, then $y^7 = (\text{cis}\,\theta)^7 = \text{cis}\,7\theta$

For $\theta = \pi/7$

$$y^7 = \cos 7\times\frac{\pi}{7} = \cos\pi = -1$$

$$y^7+1=0 \Rightarrow (y+1)(y^6-y^5+y^4-y^3+y^2-y+1)=0$$

$$\Rightarrow y+1=0 \text{ or } y^6-y^5+y^4-y^3+y^2-y+1=0$$

$$\Rightarrow y=-1 \text{ if } y^6-y^5+y^4-y^3+y^2-y+1=0$$

Dividing by y^3, we get

$$\left(y^3+\frac{1}{y^3}\right)-\left(y^2+\frac{1}{y^2}\right)+\left(y+\frac{1}{y}\right)-1=0$$

$$\therefore \left[\left(y+\frac{1}{y}\right)^3-3\left(y+\frac{1}{y}\right)\right]-\left[\left(y+\frac{1}{y}\right)^2-2\right]+\left(y+\frac{1}{y}\right)-1=0$$

Let $x = y+\frac{1}{y} \Rightarrow x = 2\cos\theta$

$$x^3-3x-(x^2-2)+x-1=0$$

$$\Rightarrow x^3-x^2-2x+1=0, \quad \text{where } x = 2\cos\theta. \quad \text{—— (1)}$$

we know that

$$\theta = \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \frac{9\pi}{7}, \frac{11\pi}{7}, \frac{13\pi}{7}$$

$$\cos\frac{13\pi}{7} = \cos\left(2\pi-\frac{\pi}{7}\right) = \cos\frac{\pi}{7}$$

$$\cos\frac{11\pi}{7} = \cos\left(2\pi-\frac{3\pi}{7}\right) = \cos\frac{3\pi}{7}$$

$$\cos\frac{9\pi}{7} = \cos\left(2\pi-\frac{5\pi}{7}\right) = \cos\frac{5\pi}{7}.$$

Hence the roots of eq (1) are $2\cos\frac{\pi}{7}, 2\cos\frac{3\pi}{7}, 2\cos\frac{5\pi}{7}$. Ans

Theorem: Expand $\sin n\theta$ and $\cos n\theta$ in powers of $\sin\theta$ & $\cos\theta$, where $n \in \mathbb{N}$.

Proof: Since $(\cos\theta+i\sin\theta)^n = (\text{cis}\,\theta)^n = \text{cis}\,n\theta$.

$$\Rightarrow \cos n\theta + i\sin n\theta = (\cos\theta+i\sin\theta)^n$$

$$= {}^nC_0\cos^n\theta + {}^nC_1\cos^{n-1}\theta(i\sin\theta) + {}^nC_2\cos^{n-2}\theta(i\sin\theta)^2 + \cdots + \cdots + {}^nC_n(i\sin\theta)^n.$$

$$= \left(\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \cdots\right) + i\left({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \cdots\right)$$

Comparing real and imaginary parts.

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \cdots \quad \text{and}$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \cdots.$$

Exp: Prove that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

Sol.:

We know that

$$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \cdots$$

$\therefore$

$$\cos 6\theta = \cos^6\theta - {}^6C_2 \cos^4\theta \sin^2\theta + {}^6C_4 \cos^2\theta \sin^4\theta - {}^6C_6 \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta \sin^2\theta + 15\cos^2\theta \sin^4\theta - \sin^6\theta.$$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

Ans

Exp (i) Express $\dfrac{\sin 6\theta}{\sin\theta}$

(ii) Expand $\cos^8\theta$ in a series of cosine of multiples of θ.

(iii) Expand $\sin^7\theta\cos^3\theta$ in a series of $\sin\theta$

(iv) Prove that $2^6\sin^7\theta = 35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta$.